# Hysteresis scaling of the field-driven first-order phase transition in the Ising model



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# Hysteresis scaling of the field-driven first-order phase transition in the Ising model

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**Abstract.** Dynamical phase transitions in the Ising model on hypercubic lattices are considered. Under a linearly swept magnetic field, the hysteresis loop that characterizes the field-driven first-order phase transition is studied carefully. Using the Glauber dynamics, we find that, in the mean-field approximation, the energy dissipation of this phase transition or the hysteresis loop area $A$ of the $M$–$H$ curve can be scaled with respect to the sweep rate $h$ of magnetic field in the form $A - A_0 \propto h^b$, $A_0 \propto (T_c - T)^a$ with $a = 2$ and $b = 2/3$. However, $b$ varies ($b < 2/3$) when fluctuations and spin correlations are taken into account. Monte Carlo simulation is used to obtain the scaling relation for $A$ in two-, three- and four-dimensional Ising models and we obtain the exponents $b = 0.36 \pm 0.06$, $0.52 \pm 0.04$ and $0.65 \pm 0.04$ respectively. These exponents are obviously different from those obtained by scaling $A$ as $A \propto h^b T^{-c}$ for any temperatures in Ising models under a sinusoidal field. Finally we point out that, in the concept of universality, field-driven first-order phase transitions in the Ising model in different dimensions belong to different universal classes due to the spin fluctuation and correlation below the Curie temperature.

## 1. Introduction

The ferromagnetic Ising model is a simple model that can exhibit a first-order phase transition (FOPT). Below the Curie temperature $T_C$, if an Ising system is subjected to a strong magnetic field $H$, the magnetization of the system may become metastable and show a discontinuity at $H = \pm H_s$, with $H_s$ the corresponding limit of metastability. The kinetic approach to metastability has been studied by Binder *et al* on this model, under the action of a static external field, by means of Monte Carlo (MC) simulations on square and cubic lattices [1, 2]. However, effects of time-dependent fields on this phase transition were not considered.

In the previous decade, the kinetics of FOPTs was extensively studied in the concept of scaling and universality [3]. Based on the scaling of the autocorrelation function and time evolution of the mean domain size of the system, the dynamical process of the magnetic system under an external field was found to have universality [2, 16]. Moreover, some research works had been concentrated on the scaling of the hysteresis loop area induced by the sweep field. Among these studies, there were some MC simulations and some phenomenological models based on Langevin dynamics. In the phenomenological model, a system with O($\mathbf{N}$) continuous symmetry was considered [4, 5]. Under the external sinusoidal field $H = H_0 \sin(\omega t)$, the hysteresis loop area $A$ induced by this perturbation was scaled as

$$\lim_{H_0\to 0} \lim_{\omega\to 0} A \propto H_0^a \omega^b.$$

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In the large-$N$ limit, $b = 1/2$ for the $(\Phi^2)^2$ model and $b = 2/3$ for the $(\Phi^2)^3$ model or mean-field theory [5–7], and the exponents were found to be independent of the profile of the periodic fields (for example sinusoidal, step-function or sawtooth fields) [4]. In MC simulations, only the Ising model was considered under a small-amplitude cosine field [8–10]. In this case, it had been pointed out that the hysteresis area could be written as the universal function for any temperatures $A \sim H_0^{\alpha} T^{-\beta} g(T^{\delta}\omega/H_0^{\gamma}) \propto H_0^{a}\omega^{b}T^{-c}$. In the low-frequency limit, $b = 0.36$, 0.45 and 0.50 for dimensionality $d = 2$, 3 and 4 respectively, while in the mean-field approximation of [8] and [9] (MFA) the authors obtained $a = 1/2$, $b = 1/2$ and $c = 1$.

We are interested in the effect of a linearly swept field with large amplitude on the Ising model, though there are no differences between the effects of a linearly swept field and a sinusoidal swept field on the $N$-vector models [4]. In our consideration, the field-driven FOPT in the Ising model is independent of the initial value of the linearly swept field $H_0$, as we keep the sweep rate $h$ constant throughout the transition and begin with the ferromagnetic spin state below $T_C$. In such a dynamical phase transition, it is advantageous that we use a linearly swept field to study the hysteresis of transition, because the hysteresis is only dependent on the competition between the relaxation time of the metastable state and the sweep rate. Therefore we can focus on the intrinsic properties of the occurrence of the hysteresis in a field-driven FOPT, namely, the fluctuation, long-range correlation and dimensionality.

The paper is organized as follows. In section 2, we present our detailed MC simulation results on two-dimensional (2D), three-dimensional (3D) and four-dimensional (4D) Ising models under linearly swept magnetic fields. The scaled relations between hysteresis area and sweep rate are given. In section 3, a series of coupled equations of magnetization and spin correlation are derived from the master equation and Glauber dynamics, and solved numerically to calculate the scaling exponents.

## 2. Monte Carlo simulations

The MC simulations are carried out on finite lattices subjected to the periodic boundary condition. Spin numbers are up to more than $10^4$ for all Ising spin systems. Effects on hysteresis due to finite size are negligible for all these spin systems when spin numbers $N > 5 \times 10^3$.

We consider a system of interacting Ising spins which are located on supercubic lattice sites. The Hamiltonian of this system is given by

$$\mathcal{H}_{Ising} = -J \sum_{\langle i,j \rangle} S_i S_j - H(t) \sum_i S_i \tag{1}$$

where the spin variables are represented by $\{S_i\}$ with $S_i = \pm 1$; $\langle i, j \rangle$ is the sum extending over all nearest-neighbour spins. $H(t)$ is a linearly swept magnetic field in the units of $k_B T/\mu_B$. $J > 0$ is the exchange interaction constant in the units of $k_B T$ ($T$ is the temperature of the spin system). When $H(t) = 0$, the spin system exhibits a second-order phase transition from a ferromagnetic state to a paramagnetic state at the Curie temperature $T_C$, $J/k_B T_C = 0.440$, 0.222 and 0.150 for 2D, 3D and 4D systems respectively [11]. Below $T_C$, there is a field-driven FOPT. The spin system might enter a metastable state characterized by $H < 0$ with magnetization $M > 0$, therefore a hysteresis loop occurs in the $M$–$H$ curve if $H(t)$ is a swept external field.

The procedure of our MC simulation is described as follows: starting from a ferromagnetic state with all spins up, under the external field $H_0 > 0$, we decrease the field linearly as $H(t) = H_0 - ht$ and obtain the $M$–$H$ curve; then at the ferromagnetic state with all spins down, under the external field $-H_0$, we increase the field as $H(t) = -H_0 + ht$. The amplitude of this swept field $H_0$ is large enough ($H_0 > |H_s|$, $H_s$ is the magnetic field of spinodal points) that the magnetization jump will occur only after the spinodal points. Consequently, the hysteresis loop of the FOPT is formed.

In our MC simulations, the sweep rate is measured by $h = \Delta H/\Delta t$. After the spin system $\{S_i\}$ under magnetic field $H$ has undergone the spin-flip Metropolis algorithm for time $\Delta t$, the output of $\{S_i\}$ is used as an initial configuration for the same algorithm at $H \pm \Delta H$. The time unit is set to be one Monte Carlo step per spin (MCS/spin). We have performed two kinds of swept field: (i) the linearly swept field and (ii) the step-function swept field. In the first case $\Delta t$ is always maintained to be 1 MCS/spin; $h$ can be modified by changing the value of $\Delta H$. In the second kind of magnetic field, $\Delta H$ is fixed to be a constant and $h$ varies if the holding time $\Delta t$ (MCS/spin) is different. The magnetization of the system is calculated from $M = \langle \Sigma S_i \rangle / N$, where $\langle\rangle$ represents the thermal average. Under a field of type (i) this average is taken five to ten times; while under a field of type (ii) $M$ is determined by a coarse-grained average [12]. We have checked several times that at a sufficiently low swept rate $h$ the $M$–$H$ curves are the same. Therefore we only present the MC results under condition (i).

The energy dissipation of an FOPT can be calculated from the hysteresis area of the $M$–$H$ curve: $A = \oint M \mathrm{d}H$. The integral is evaluated by an adaptive recursive Newton–Cotes eight-panel rule. Cubic interpolation is used to adapt to the integral of $M$–$H$ data obtained from MC simulations. In this kind of numerical method, the accuracy of $A$ is found to be less than 1%.

### *2.1. Scaling of the hysteresis in the 4D Ising model*

MC simulation of the 4D Ising model is carried out in the Ising spin system with spin numbers $N = 15^4$. Figure 1(a) is the hysteresis loops at various temperatures below $T_C$. At a fixed temperature, the loop area $A$ decreases with decreasing $h$ and will reach a value of $A_0$, $A_0$ tends to zero when $T$ is near $T_C$. We can then scale the hysteresis area as

$$A = A_0 + f(T)H^b \qquad A_0 \propto (T_C - T)^a \qquad f(T) = f(T_C - T). \tag{2}$$

At temperatures above $T_C$, there is also a hysteresis loop but $A$ decreases to zero at small $h$, and it can be scaled in another form:

$$A = g(T)h^b \qquad g(T) = g(T - T_C) \tag{3}$$

the exponent $b$ can be fitted in expressions (2) or (3) by means of least-squares fitting. The fitting results are shown in figure 2(a) for various temperatures. It can be found that, at small sweep rates, the exponents $b$ are the same within calculation error for any temperatures below the critical temperature: $b = 0.65 \pm 0.04$. Above $T_C$, the relationship between $A$ and $h$ also has a scaling exponent if $1.2T_C > T > T_C$. However, this exponent $b = 0.94 \pm 0.06$ is different from that below $T_C$. The temperature scaling exponent $a$ for static hysteresis area $A_0$ is valid only for the hysteresis that is obtained at temperatures below $T_C$, and we obtain $a = 1.82 \pm 0.04$.

The scaling exponent $b$ in the 4D Ising model is the same as the mean-field approximation (MFA) results: $b = 2/3$ ($T < T_C$) and $b = 1$ ($T > T_C$), which will be evaluated in section 3.

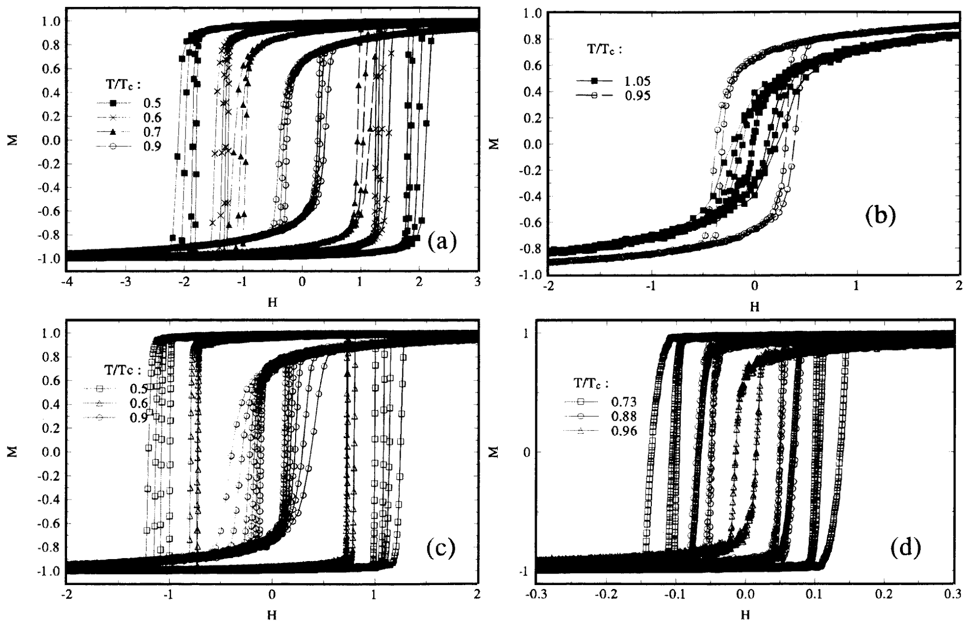


**Figure 1.** MC simulation results of Ising models: some $M$–$H$ hysteresis loops under external fields with various sweep rates. The sweep rates $h$ (from inner to outer loops at the same transition temperature) are as follows. (a), (b) 4D Ising model: $h = 0.01$, 0.02, 0.04, 0.08 ($T = 0.5T_C$); $h = 0.005$, 0.01, 0.02, 0.04 ($T = 0.6T_C$); $h = 0.01$, 0.02, 0.04 ($T = 0.7T_C$); $h = 0.01$, 0.02, 0.04 ($T = 0.9T_C$); $h = 0.02$, 0.04 ($T = 0.95T_C$); $h = 0.02$, 0.05, 0.1, 0.2 ($T = 1.05T_C$). (c) 3D Ising model: $h = 1 \times 10^{-3}$, $2 \times 10^{-3}$, $4 \times 10^{-3}$, $8 \times 10^{-3}$ ($T = 0.5T_C$); $h = 5 \times 10^{-4}$, $1 \times 10^{-3}$, $2 \times 10^{-3}$ ($T = 0.6T_C$); $h = 5 \times 10^{-4}$, $1 \times 10^{-3}$, $2 \times 10^{-3}$, $2.5 \times 10^{-2}$, $5 \times 10^{-2}$ ($T = 0.9T_C$). (d) 2D Ising model: $h = 10^{-4}$, $2 \times 10^{-4}$, $5 \times 10^{-4}$ ($T = 0.73T_C$); $h = 5 \times 10^{-5}$, $1 \times 10^{-4}$, $1.5 \times 10^{-4}$, $2 \times 10^{-4}$ ($T = 0.88T_C$); $h = 1 \times 10^{-4}$ ($T = 0.95T_C$).

### *2.2. 2D and 3D Ising models*

The cubic lattice and the square lattice for MC simulations have sizes of $30^3$ and $100^2$ respectively. The hysteresis loops for the field-driven FOPT are shown in figure 1(c) for the 3D Ising system and figure 1(d) for the 2D Ising system. The scaled relationships between loop areas and sweep rates at various temperatures are shown in figure 2(b) and (c) respectively. They can also be scaled as the expressions described in equations (2) and (3).

For the 3D Ising system below $T_C$, the exponents in expression (2) are $b = 0.52 \pm 0.04$ and $a = 1.78 \pm 0.04$. However, there is no definite value for the exponent in expression (3). We find that the parameter $b$ increases if the temperature $T(>T_C)$ increases and it does not seem to be a constant. Here we would like to point out the differences between expressions (2) and (3). Expression (2) is a universal scaling relation for the field-driven FOPT in the Ising model. However, above $T_C$, the hysteresis loop results from the magnetization process of the spin system under the swept magnetic field, rather than the FOPT. Therefore expression (3) does not have non-zero static hysteresis and a scaling exponent. We can also find out the differences between our MC results and those in [8–10]. The cosine field the authors of [8–10] used was so small that transition occurred just as $H = 0^+$ or $H = 0^-$; the system did not enter the metastable state and the hysteresis could only be scaled as expression (3).

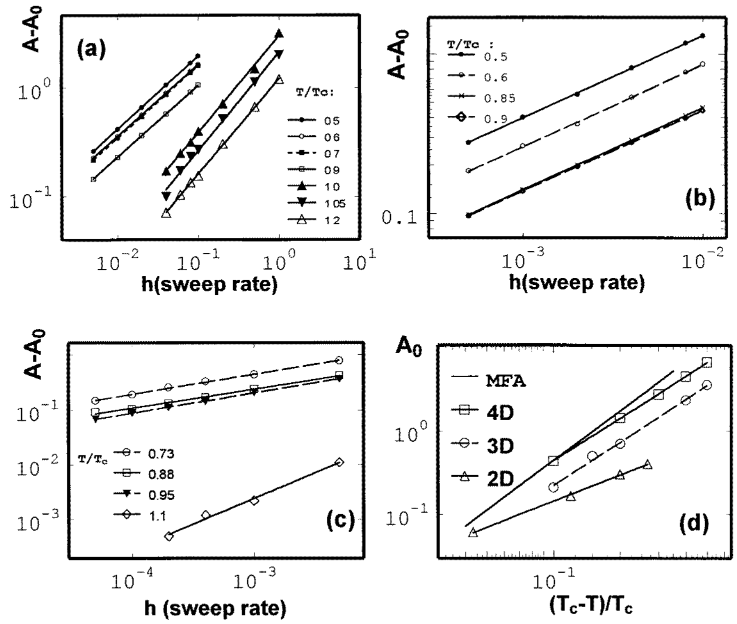


**Figure 2.** Fitting of the hysteresis loop areas with respect to the sweep rate of magnetic field at different temperatures in the scaling expressions (2) and (3), in (a) two-, (b) three- and (c) four-dimensional Ising models. The static hysteresis loop areas $A_0$ at different transition temperatures are given in (d).

In the 2D Ising system, the exponents in scaling expression (2) are $b = 0.36 \pm 0.06$ and $a = 1.04 \pm 0.08$. Similar to the 3D Ising system, the hysteresis loop area cannot be scaled with respect to the sweep rate if the hysteresis does not characterize the field-driven FOPT.

### *2.3. Summary*

The exponents in scaling expressions (2) and (3) are listed for 2D, 3D and 4D systems in table 1. In 2D and 3D systems, no scaling exponents are found if the systems do not exhibit a field-driven FOPT. The scaling exponents $a$ for all dimensions are compared with the MFA result and they are all shown in figure 2(d).

## 3. Hysteresis scaling based on the Glauber dynamics of the Ising model

The kinetic Ising model can be obtained from the master equation describing the evolution of the spin variables $\{S_i\}$. Now the Hamiltonian of this spin system is described by equation (1) and an arbitrary state of the system is represented by a set of probability $P(\{S_i\}, t)$, where $\{S_i\} = \{S_1, \ldots, S_j, \ldots, S_N\}$. Following the single-spin flip dynamics (Glauber dynamics), the time evolution of the system may be governed by the master equation:

$$\frac{\mathrm{d}}{\mathrm{d}t} P(S_1, \ldots, S_j, \ldots, S_n, t) = -\left[\sum_{j=1}^{N} W_j(S_j)\right] P(S_1, \ldots, S_j, \ldots, S_n, t) + \sum_{j=1}^{N} W_j(-S_j) P(S_1, \ldots, -S_j, \ldots, S_n, t) \tag{4}$$

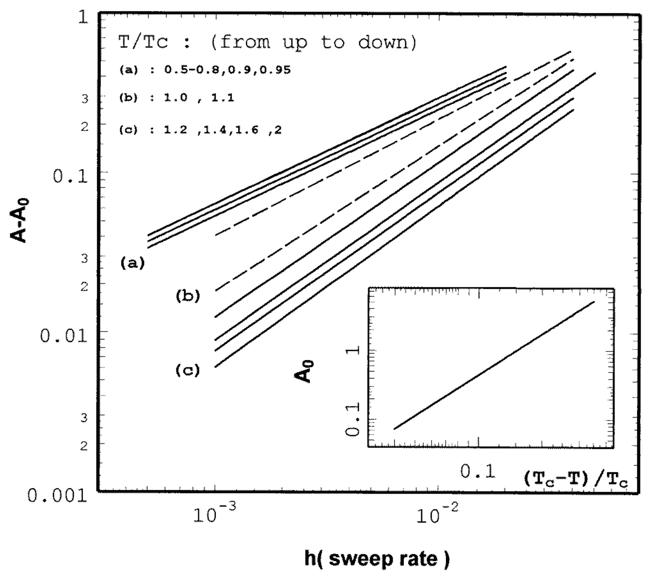


**Figure 3.** Hysteresis scaling by MFA. The scaling exponent $b$ is different below and above the critical temperature. The inset shows the static hysteresis loop area $A_0$ at different transition temperatures. $A_0 = 0$ for $T \geqslant T_c$.

where $W_j(S_j)$ is the transition probability from spin configuration $\{S_1, \ldots, S_j, \ldots, S_N\}$ to $\{S_1, \ldots, -S_j, \ldots, S_N\}$, and it must be determined from the detailed balance conditions:

$$p_e(S_1, \ldots, S_j, \ldots, S_n) W_j(S_j) = p_e(S_1, \ldots, -S_j, \ldots, S_n) W_j(-S_j) \tag{5}$$

with $p_e = \exp[-\mathcal{H}(\{S_i\})/k_B T]/Z$, in Glauber dynamics [13]

$$W_j(S_j) = (\alpha/2)[1 - S_j \tanh(E(S_j))] \qquad E(S_j) = J \sum_i S_i + H. \tag{6}$$

Let $m(t) = \mu_1(j,t) = \langle S_j \rangle$, $\mu_2(ij,t) = \langle S_i S_j \rangle$, $\mu_n(ij\ldots n,t) = \langle S_i S_j \ldots S_n \rangle$, then the master equation (4) is reduced to the following coupled equations ($n = 1, 2, 3, \ldots$):

$$\frac{\mathrm{d}}{\mathrm{d}t}\mu_n = -\alpha\mu_n + \alpha \sum_{j=1}^{n} \left\langle \left( \prod_{i \neq j}^{n} S_i \right) \tanh E(S_j) \right\rangle. \tag{7}$$

### *3.1. The mean-field approximation*

In the thermodynamic limit $N \to \infty$, and with $\mu_n = 0$ except $\mu_1(j,t) = \langle S_j \rangle = m(t)$ independent of site $j$, we have equation (7) with $\alpha = 1$:

$$\frac{\mathrm{d}m(t)}{\mathrm{d}t} = -m(t) + \tanh[(m(t) + H(t))/T] \tag{8}$$

where $H$ and $T$ are dimensionless in the units of $(qJ)$ and $(qJ/k_B)$ respectively and $q$ is the numbers of nearest-neighbour sites. The critical temperature at $H = 0$ is $T_c = 1$. At $T < T_c$, the spinodal point $(H_s, m_s)$ on the $m$–$H$ curve can be obtained by setting the left-hand side of equation (8) to zero, and we have $H_s = T \ln(1 + \sqrt{1-T}) - T \ln T - \sqrt{1-T}$, $m_s = \sqrt{1-T}$. Under the linearly swept field $H(t) = H_0 - ht$ with $H_0 \gg |H_s|$, (8) is solved numerically using fourth and fifth Runge–Kutta formulas. The hysteresis loop area

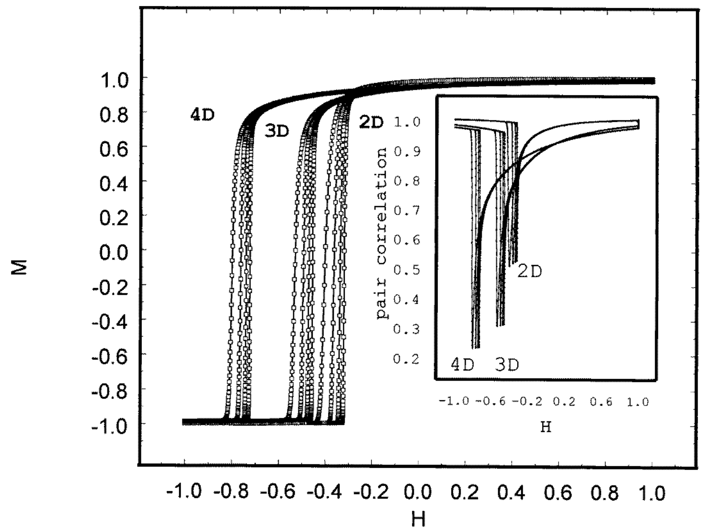


**Figure 4.** Magnetization curves at different dimensions when fluctuations are considered. $h = 5 \times 10^{-4}$, $1 \times 10^{-3}$, $2 \times 10^{-3}$, $4 \times 10^{-3}$ for 2D (at $k_B T/J = 0.8$); 3D (at $k_B T/J = 1.5$) and 4D (at $k_B T/J = 2$). Inset, the pair correlation function near the transition.

can be scaled with respect to the sweep rate $h$ as expressions (2) and (3) for $T$ below and above $T_c$ respectively: the results are shown in figure 3. We have the exponents $b = 2/3$ ($T < T_c$) and $b = 1$ ($T > T_c$). The static hysteresis $A_0$ at $T < T_c$ is shown in the figure inserted in figure 3 and the scaling exponent $a$ is given in table 1.

**Table 1.** Scaling exponents of the field-driven FOPT in Ising models in different dimensions ($H_0$ is the amplitude of the sweep field).

| | 2D ($H_0 = 1$) | 3D ($H_0 = 4$) | 4D ($H_0 = 5$) | MFA |
|---|---|---|---|---|
| $b$ (above $T_C$) | — | — | $0.94 \pm 0.06$ | 1 |
| $b$ (below $T_C$) | $0.36 \pm 0.06$ | $0.52 \pm 0.04$ | $0.65 \pm 0.04$ | 2/3 |
| $a$ | $1.04 \pm 0.08$ | $1.78 \pm 0.04$ | $1.82 \pm 0.04$ | 2 |

### *3.2. The second-order approximation*

Now we consider the pair correlation between nearest-neighbour $\mu_2(ij, t) = c(t) = \langle S_i S_j \rangle$, as the second-order approximation in equation (7). Using the fluctuation approximation derived by Mamada *et al* [15], equation (7) is then reduced to a coupled set of equations:

$$\frac{\mathrm{d}m(t)}{\mathrm{d}t} = -m(t) + \frac{1}{2^{q+1}} \sum_{n=0}^{q} C_q^n \tanh\left[\left(H + \frac{q-2n}{2}\right)\Big/ T\right] F(m(t), c(t))$$

$$\frac{\mathrm{d}c(t)}{\mathrm{d}t} = -2c(t) + \frac{1}{2^{q}} \sum_{n=0}^{q} C_q^n \frac{q-2n}{q} \tanh\left[\left(H + \frac{q-2n}{2}\right)\Big/ T\right] F(m(t), c(t)) \tag{9}$$

$F(m, c) = (1 + m(t))^{-q+1}(1 + 2m(t) + c(t))^{q-n}(1 - c(t))^n + (1 - m(t))^{-q+1}(1 - 2m(t) + c(t))^n(1 - c(t))^{q-n}$.

$m(t)$ and $c(t)$ are given in figure 4 by solving (9) numerically. In 2D, 3D and 4D Ising spin systems, the nearest-neighbour numbers in these hypercubic lattices are $q = 4$, 6, 8 respectively. Below $T_c$, the scaling of the hysteresis can be written as expression (2); the method for determining the sweep rate scaling exponent $b$ is analogous to the MFA. We obtain $b = 0.58$, 0.61 and 0.66 for 2D, 3D and 4D Ising systems respectively.

### *3.3. Discussion*

Comparing the exponents $b$ obtained by the fluctuation approximation with those obtained by the MFA, we find that $b$ is smaller than 2/3 at lower dimension because of the fluctuation and correlation of spins, while in 4D $b \sim 2/3$ because the strong fluctuations are negligible. However, the exponents $b$ calculated by this second-order approximation are still far larger than MC results in 2D and 3D. This is because, below the critical temperature, correlation of spins is not restricted to the pair correlation and, among the nearest-neighbour sites, correlation of spins in some large clusters is evident below $T_c$ [16]. We have considered a higher-order approximation in equation (7) by taking more $\mu_n$ into account, but have not obtained the optimized values of scaling exponents, compared with MC results. The time-dependent renormalization group by means of the Migdal–Kananoff approximation (bond-moving technique) might help in this kinetic Ising model.

## 4. Conclusions

The hysteresis of a field-driven first-order phase transition in the Ising model is studied. We have considered the effects of linearly swept field on the energy dissipation of the FOPT. In Monte Carlo simulations, we find that these energy dissipations can be scaled with respect to the sweep rates of external field. The scaling exponent $b$ is independent of the temperature at which the FOPT occurs. The dimensional effect has been studied for the Ising system in different spatial dimensions: we obtain the sweep rate scaling exponents to be 0.36, 0.52 and 0.65 for $d = 2, 3$ and 4 respectively. Therefore we conclude that the sweep rate scaling for the energy dissipation of the field-driven FOPT is universal for the scalar model, and $d$-dimensional ($2 \leqslant d \leqslant 4$) Ising models belong to different universal classes. Using the Glauber dynamics, we compare the MC results with numerical results obtained from the kinetic Ising model by means of the fluctuation approximation and MFA. Different scaling exponents in $d$-dimensional Ising models are contributed to by the strong fluctuation and correlation of spins below the critical temperature.

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